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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

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N° 2917

Juin 1996

PROGRAMME 2

Rapport de recherche

1996

ISSN 0249-6399

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Abstract

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Sommes d'Euler et représentations intégrales

Résumé

Cet article développe une approche à l'évaluation de sommes d'Euler qui font intervenir des nombres harmoniques linéairement ou polynomialement. Nous donnons des formules explicites pour certaines classes de sommes d'Euler en termes de valeurs de la fonction Zeta de Riemann à des entiers positifs. Cette approche est fondée sur des représentations intégrales et des calculs de résidus.

EULER SUMS AND CONTOUR INTEGRAL REPRESENTATIONS

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June 18, 1996

Abstract. This paper develops an approach to the evaluation of Euler sums involving harmonic numbers either linearly or nonlinearly. We give explicit formulæ for certain classes of Euler sums in terms of values of the Riemann zeta function at positive integers. The approach is based on simple contour integral representations and residue computations.

1. Introduction

Harmonic numbers and their generalizations are classically defined by

$$H_n \equiv H_n^{(1)} := \sum_{j=1}^{n} \frac{1}{j}, \qquad H_n^{(r)} := \sum_{j=1}^{n} \frac{1}{j^r}. \tag{1}$$

The subject of this paper is *Euler sums* that are the infinite sums whose general term is a product of harmonic numbers of index n and a power of n^{-1}. It has been discovered in the course of years that many Euler sums admit expressions involving finitely the “zeta values”, that is to say values of the Riemann zeta function,

$$\zeta(s) := \sum_{j=1}^{\infty} \frac{1}{j^s} \tag{2}$$

at the positive integers. Typical evaluations to be discussed here are

$$\begin{array}{ll}
(a) & \displaystyle\sum_{n\geq 1} \frac{H_n}{n^2} = 2\zeta(3), \qquad \sum_{n\geq 1} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \qquad \sum_{n\geq 1} \frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3) \\
(b) & \displaystyle\sum_{n\geq 1} \frac{H_n^{(2)}}{n^4} = \zeta(3)^2 - \frac{1}{3}\zeta(6) \\
(c) & \displaystyle\sum_{n\geq 1} \frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7) \\
(d) & \displaystyle\sum_{n\geq 1} \frac{(H_n)^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{12}\zeta(3)\zeta(4) \\
(e) & \displaystyle\sum_{n\geq 1} \frac{(H_n)^3}{n^4} = \frac{231}{16}\zeta(7) - \frac{51}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5) \\
(f) & \displaystyle\sum_{n\geq 1} \frac{(H_n)^4}{(n+1)^3} = \frac{185}{8}\zeta(7) - \frac{43}{2}\zeta(3)\zeta(4) + 5\zeta(2)\zeta(5),
\end{array} \tag{3}$$

as well as relations like

$$\sum_{n\geq 1}\frac{(H_n)^3}{n^5}-\frac{11}{4}\sum_{n\geq 1}\frac{H_n^{(2)}}{n^6}=\frac{469}{32}\zeta(8)-16\zeta(3)\zeta(5)+\frac{3}{2}\zeta(2)\zeta(3)^2. \tag{4}$$

Euler started this line of investigation in the course of a correspondence with Goldbach beginning in 1742 (see [2, p. 253] for a discussion) and he was the first to consider the *linear* sums,

$$S_{p,q}:=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}. \tag{5}$$

Euler, whose investigations were to be later completed by Nielsen [13], discovered that the linear sums have evaluations in terms of zeta values in the following cases: $p=1$; $p=q$; $p+q$ odd; $p+q$ even but with the pair (p,q) being restricted to a finite set of so-called "exceptional" configurations $\{(2,4),(4,2)\}$. Of these cases, the one corresponding to $p=q$ is obvious given the symmetry relations

$$S_{p,q}+S_{q,p}=\zeta(p)\zeta(q)+\zeta(p+q), \tag{6}$$

while the other ones correspond to essentially nontrivial identities of which (a,b,c) of Eq. (3) are typical. Rather extensive numerical search for linear relations between linear Euler sums and polynomials in zeta values [1] strongly suggest that Euler found all the possible evaluations of linear sums.

The next object of interest is the *nonlinear* sums that involve products of at least two harmonic numbers. Let $\varpi=(\varpi_1,\ldots,\varpi_k)$ be a partition of integer p into k summands, so that $p=\varpi_1+\cdots+\varpi_k$ and $\varpi_1\leq\varpi_2\leq\cdots\leq\varpi_k$. The Euler sum of index ϖ,q is defined by

$$S_{\varpi,q}=\sum_{n=1}^{\infty}\frac{H_n^{(\varpi_1)}H_n^{(\varpi_2)}\cdots H_n^{(\varpi_k)}}{n^q}, \tag{7}$$

the quantity $q+\varpi_1+\cdots+\varpi_k$ being called the *weight* and the quantity k being the *degree*. As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^22^35,q}=S_{112225,q}=\sum_{n=1}^{\infty}\frac{(H_n)^2(H_n^{(2)})^3H_n^{(5)}}{n^q}.$$

In the past, a few basic nonlinear sums have been evaluated thanks to their relations to the Eulerian beta integrals or to polylogarithms [7]. Recently, a detailed numerical search conducted by Bailey, Borwein, Girgensohn [1] has revealed the existence of many surprising evaluations like the cubic or quartic formulæ of (3). Some of these have since received a due proof and for instance the paper [4] gives explicit formulæ for

$$S_{1^2,q}=\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^q}$$

whenever the weight $q+2$ is odd (see (d) of Eq. (3)), and an explicit reduction to $S_{2,q}$ when the weight is even.

The situation regarding explicit evaluations of Euler sums is at first sight rather puzzling. Some evaluations appear to generalize and form an infinite class —like $S_{1^2,q}$ above— while others seem to vanish mysteriously as soon as the weight exceeds a certain threshold. For instance, no finite formula in terms of zeta values is likely to exist for the cubic sums $S_{1^3,q}$ or the quartic sums $S_{1^4,q}$ of an odd weight exceeding 10 while $S_{1^3,4}, S_{1^4,3}$ of Eq. (3) or even the septic $S_{1^7,2}$ do reduce to zeta values [1]. This suggests the existence of both "general" classes of evaluations and "exceptional" evaluations.

A recent approach exemplified by the work of Hoffman [9] and Zagier [16] sheds a new light on these phenomena. It is based on considering the *multiple zeta* functions defined by

$$\zeta(a_1, a_2, \ldots, a_\ell) := \sum_{n_1 < n_2 < \cdots < n_\ell} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_\ell^{a_\ell}}, \tag{8}$$

where $a_1 + \cdots + a_\ell$ is called the *weight* and ℓ is the multiplicity. Every Euler sum of weight w and degree k is clearly a $\mathbb{Q}$-linear combination of multiple zeta values (*i.e.*, values of multiple zeta functions at integer arguments) of weight w and multiplicity at most $k+1$. In other words, multiple zeta values are "atomic" quantities into which Euler sums decompose. Consequently, a complete model for the linear relations involving the multiple zeta values would yield a full decision procedure for determining whether any particular Euler sum admits a complete evaluation in terms of (single) zeta values.

A conjecture of Zagier to be discussed later states that the the dimension d_w of the $\mathbb{Q}$-linear space generated by the 2^{w-2} multiple zeta values of weight w increases roughly like 1.32^w. In contrast the number μ_w of weight-homogeneous monomials in zeta values of weight w is much smaller asymptotically, being only $e^{O(\sqrt{w})}$. Thus, *a priori*, only a small fraction of quantities expressible in terms of multiple zetas should reduce to polynomials in (single) zeta values. However, initially, the difference $d_w - \mu_w$ is small and even equal to 0 for some of the low weights, $\{3, 4, 5, 6, 7, 9\}$. As a consequence, any Euler sum of odd weight ≤ 9 *must* reduce to zeta values. The multiple zeta model therefore explains well the presence of exceptional evaluations of Euler sums that appear in this perspective to be unavoidable artefacts of low weight.

A characteristic aspect of the multiple zeta model is that it may predict *relations* but does not in general provide *explicit formulæ*. This is where we fit in. Our approach is based on contour integral representations. It is directed at Euler sums that are particular "non-atomic" combinations of multiple zeta values, having almost complete symmetry. When applicable, this approach does not require inverting collections of linear relations, which may be rather difficult to do for a whole class of sums as exemplified by the works of Borwein *et al.* [4, 5].

We mention finally that Euler sums and multiple zetas have connections with many branches of mathematics, see especially Zagier's discussion in [16]. Broadhurst (see [5]) encountered them in relation with Feynman diagrams and associated knots in perturbative quantum field theory. They also surface occasionally in combinatorial mathematics: the Euler evaluation (a) of Eq. (3) serves to analyze the distribution of node degrees in quadtrees [8, 10] while alternating Euler sums make an appearance in the analysis of lattice reduction algorithms [6].

2. General summations

Contour integration is a classical technique for evaluating infinite sums by reducing them to a finite number of residue computations. For instance the identity

$$2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi} - e^{-\pi}} - 1$$

derives from a residue computation of the integral

$$\frac{1}{2i\pi} \int \frac{\pi}{\sin \pi s} \frac{ds}{s^2+1}$$

over a circle centred at the origin and whose radius is taken arbitrarily large. The residues at the poles $s = \pm n$ with $n \neq 0$ generate the left-hand side of the equality, while the poles at $s = 0, \pm i$ yield the explicit form appearing on the right.

This summation mechanism is formalized by a lemma that goes back to Cauchy and is nicely developed throughout Lindelöf's book [11]. We define a *kernel function* $\xi(s)$ by the two requirements: $\xi(s)$ is meromorphic in the whole complex plane; $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|z| = \rho_k$ with $\rho_k \to +\infty$.

Lemma 1 (Cauchy, Lindelöf). *Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then*

$$\sum_{\alpha \in O} \operatorname{Res}(r(s)\xi(s))_{s=\alpha} = -\sum_{\beta \in S} \operatorname{Res}(r(s)\xi(s))_{s=\beta} \tag{9}$$

where S is the set of poles of $r(s)$ and O is the set of poles of $\xi(s)$ that are not poles of $r(s)$. There, $\operatorname{Res}(h(s))_{s=\lambda}$ *denotes the residue of $h(s)$ at $s = \lambda$.*

Proof. It suffices to apply the residue theorem to

$$\frac{1}{2i\pi} \int_{(\infty)} r(s)\xi(s)\, ds,$$

where $\int_{(\infty)}$ denotes integration along large circles, i.e., the limit of integrals $\int_{|s|=\rho_k}$. □

This formula does have the character of a summatory formula since the set O of poles of an irrational kernel $\xi(s)$ (called the "ordinary poles") is infinite, while the set of S of poles of a rational function $r(s)$ (the "special poles") is necessarily finite. We also define the *special residue sum* to be the finite sum

$$\mathcal{R}[\xi(s)r(s)] := \sum_{\alpha \in \mathcal{S} \cup \{0\}} \operatorname{Res}(\xi(s)r(s))_{s=\alpha}. \tag{10}$$

The amalgamation of 0 to the special poles is just a notational convenience dictated by the frequent need to isolate 0 in summatory formulæ. Then (9) is rephrased as

$$\sum_{\alpha \in O \setminus \{0\}} \operatorname{Res}(r(s)\xi(s))_{s=\alpha} = -\mathcal{R}[\xi(s)r(s)]. \tag{11}$$

Let $[(s-\lambda)^r]h(s)$ denote the coefficient of $(s-\lambda)^r$ in the Laurent expansion of $h(s)$ at $s = \lambda$. Residues are Laurent coefficients, and as such are computable like Taylor coefficients since

$$\operatorname{Res}(h(s))_{s=\lambda} = [(s-\lambda)^{-1}]h(s) = [(s-\lambda)^{r-1}](s-\lambda)^r h(s),$$

if r is the order of the pole of $h(s)$ at $s = \lambda$. In other words, the special residue sum is always determined by a few Taylor series expansions taken at a finite collection of points.

We make here an essential use of kernels involving the ψ function. The ψ function [15] is the logarithmic derivative of the Gamma function,

$$\psi(s) = \frac{d}{ds} \log \Gamma(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} [\frac{1}{n} - \frac{1}{n+s}] \tag{12}$$

and it satisfies the complement formula

$$\psi(s) - \psi(-s) = -\frac{1}{s} - \pi \cot \pi s,$$

as well as an expansion at $s=0$ that involves the zeta values:

$$\psi(s)+\gamma = -\frac{1}{s}+\zeta(2)s-\zeta(3)s^2+\cdots. \tag{13}$$

From classical expansions and the properties just recalled of the ψ function, one has at an integer n:

(14)

$$\begin{array}{llll}
\pi\cot\pi s & \underset{s\to n}{=} & \dfrac{1}{s-n}-2\displaystyle\sum_{k=1}^{\infty}\zeta(2k)(s-n)^{2k-1}, & \\
\dfrac{\pi}{\sin\pi s} & \underset{s\to n}{=} & (-1)^n\left(\dfrac{1}{(s-n)}+2\displaystyle\sum_{k=1}^{\infty}(1-2^{1-2k})\zeta(2k)(s-n)^{2k-1}\right), & \\
\psi(-s)+\gamma & \underset{s\to n}{=} & \dfrac{1}{s-n}+H_n+\displaystyle\sum_{k=1}^{\infty}[(-1)^kH_n^{(k+1)}-\zeta(k+1)](s-n)^k, & n\geq 0\\
\psi(-s)+\gamma & \underset{s\to -n}{=} & H_{n-1}+\displaystyle\sum_{k=1}^{\infty}[H_{n-1}^{(k+1)}-\zeta(k+1)](s+n)^k, & n>0\\
\dfrac{\psi^{(p-1)}(-s)}{(p-1)!} & \underset{s\to n}{=} & \dfrac{1}{(s-n)^p}\left(1+(-1)^p\displaystyle\sum_{i\geq p}\binom{i-1}{p-1}[\zeta(i)+(-1)^iH_n^{(i)}](s-n)^i\right), & n\geq 0, p>1\\
\dfrac{\psi^{(p-1)}(-s)}{(p-1)!} & \underset{s\to -n}{=} & (-1)^p\displaystyle\sum_{i\geq 0}\binom{p-1+i}{p-1}[\zeta(p+i)-H_{n-1}^{(p+i)}](s+n)^i, & n>0, p>1\\
\dfrac{1}{s^q} & \underset{s\to n}{=} & \displaystyle\sum_{j\geq 0}(-1)^j\binom{q+j-1}{q-1}\dfrac{(s-n)^j}{n^{q+j}}, & n\neq 0, q\in\mathbb{Z}_+.
\end{array}$$

Each of these functions, or any of its derivatives, is $O(|s|^\epsilon)$ on circles of radius $n+1/2$ (with n a positive integer) centred at the origin. Consequently, any polynomial form in

$$\pi\cot\pi s, \qquad \frac{\pi}{\sin\pi s}, \qquad \psi^{(j)}(\pm s) \tag{15}$$

is itself a kernel function with poles at a subset of the integers. The purpose of this paper is precisely to investigate the power of such kernels in connection with summatory formulæ and Euler sums.

We shall impose throughout two conditions on the rational function $r(s)$,

$$\begin{cases} (i) & r(s) \text{ is } O(s^{-2}) \text{ at infinity,}\\ (ii) & r(s) \text{ has no pole in } \{\ldots,-2,-1\}\cup\{1,2,\ldots\}, \end{cases} \tag{16}$$

where the first one is necessary for absolute convergence of the sums and the second one is only a minor technical requirement. A direct use of the kernels of (15) then yields the summatory formulæ,

$$\begin{array}{lll}
\displaystyle\sum_{n=1}^{\infty}r(n) & = & -\mathcal{R}[r(s)(\psi(-s)+\gamma)]\\
\displaystyle\sum_{n=1}^{\infty}(r(n)+r(-n)) & = & -\mathcal{R}[r(s)\pi\cot\pi s]\\
\displaystyle\sum_{n=1}^{\infty}(-1)^n(r(n)+r(-n)) & = & -\mathcal{R}[r(s)\dfrac{\pi}{\sin\pi s}],
\end{array} \tag{17}$$

that are classical. The kernels are $\psi(-s)+\gamma$, $\pi\cot\pi s$, $\pi/\sin\pi s$, as is apparent from the argument of the special residue sum. Clearly, the last two equalities in (17) become trivial if the rational function $r(s)$ is odd. (Such parity phenomena surface recurrently in Euler sums evaluation.)

A more interesting kernel is $(\psi(-s)+\gamma)^2$ whose residues at the positive integer generate harmonic numbers since

$$(\psi(-s)+\gamma)^2 \underset{s\to n}{\sim} \frac{1}{(s-n)^2} + 2H_n\frac{1}{s-n} + \cdots .$$

In that case, under the conditions of (16), we find

$$2\sum_{n=1}^{\infty} r(n)H_n + \sum_{n=1}^{\infty} r'(n) = -\mathcal{R}[r(s)(\psi(-s)+\gamma)^2] \tag{18}$$

as results directly from the singular expansion (14) of the kernel. Thus, by (17) and (18), any sum whose general term is the product of the harmonic number H_n and a rational function $r(n)$ reduces to a finite combination of values of the ψ function and its derivatives taken at a finite set of points. Instantiating this treatment to the class of functions $r(s)=n^{-q}$, with q an integer ≥ 2 produces a formula already known to Euler.

Theorem 1 (Euler). *For integer $q\geq 2$,*

$$S_{1,q} \equiv \sum_{n=1}^{\infty}\frac{H_n}{n^q} = (1+\frac{q}{2})\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k). \tag{19}$$

Proof. A direct consequence of the summatory formula (18) and the expansion (13). □

Special values are given in (a) of Eq. (3).

The treatment just developed of the simplest Euler sums is typical. For the case when $r(s)=n^{-q}$, only one residue needs to be determined, and the residue computation is strictly equivalent to a coefficient extraction. Given that the kernels employed throughout this paper are polynomials in ψ and related trigonometric functions, the expressions obtained are invariably weight-homogeneous convolutions of zeta values. In addition, the degree of the kernel employed (that is itself suggested by the nature of each Euler sum considered) dictates the multiplicity of the convolution formulæ that are obtained by this process.

3. Linear Euler sums

Nielsen [13], elaborating upon Euler's work, proved by a method based on partial fraction expansions that every linear sum $S_{p,q}$ whose weight $p+q$ is odd is expressible as a polynomial in zeta values. To give an idea of the method [13, p. 50], we show that $S_{1,2}=2\zeta(3)$, an equality expressed in terms of double zetas as $\zeta(1,2)=\zeta(3)$. We have

$$\begin{aligned}\zeta(1,2) &= \sum_{0<a<n}\frac{1}{an^2} = \sum_{0<a<n}\frac{1}{(n-a)n^2}\\ &= \sum_{0<a<n} -\frac{1}{an^2} + \left[\frac{1}{a^2(n-a)} - \frac{1}{a^2 n}\right]\\ &= -\zeta(1,2) + \sum_{0<a}\frac{1}{a^2}\left[\left(\frac{1}{1}-\frac{1}{a+1}\right)+\left(\frac{1}{2}-\frac{1}{a+2}\right)+\cdots\right],\end{aligned}$$

where the second line results from a partial fraction expansion and the third line from series rearrangements. The last sum telescopes and yields

$$\zeta(1,2) = -\zeta(1,2) + [\zeta(1,2)+\zeta(3)].$$

This example is typical. In general the method [4, 13] provides linear relations between the $S_{p,q}$ of the same weight and quadratic forms in zeta functions, from which a constructive (but not clearly explicit) reduction to zeta values can be derived. Borwein, Borwein, and Girgensohn [4] have succeeded in "inverting" the Euler-Nielsen relations by means of combinatorial matrix decompositions. We show here how to rederive directly the explicit evaluations of [4].

Theorem 2 (Euler, Borwein et al.). *For an odd weight $m = p+q$, the linear sums are reducible to zeta values,*

$$\sum_{n=1}^{\infty} \frac{H^{(p)}(n)}{n^q} = \zeta(m)\left[\frac{1}{2} - \frac{(-1)^p}{2}\binom{m-1}{p} - \frac{(-1)^p}{2}\binom{m-1}{q}\right] + \frac{1-(-1)^p}{2}\zeta(p)\zeta(q)$$
$$+ (-1)^p \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \binom{m-2k-1}{q-1}\zeta(2k)\zeta(m-2k) + (-1)^p \sum_{k=1}^{\lfloor \frac{q}{2} \rfloor} \binom{m-2k-1}{p-1}\zeta(2k)\zeta(m-2k),$$

where, in the formula, $\zeta(1)$ should be interpreted as 0 wherever it occurs.

Proof. In the context of this paper, the theorem results from applying the kernel

$$\frac{1}{2}\pi\cot(\pi s)\frac{\psi^{(p-1)}(-s)}{(p-1)!},$$

to the base function $r(s) = s^{-q}$. The only singularities are poles at the integers. At $-n$, where n is a positive integer, the pole is simple and the residue is

$$\frac{(-1)^m}{2n^q}\left(\zeta(p) - H_n^{(p)} + \frac{1}{n^p}\right).$$

At $+n$, where n is a positive integer, the pole has order $p+1$ and the residue is

$$\frac{1}{2n^q}(H_n^{(p)} - \zeta(p)) + (-1)^p\binom{m-1}{p}\frac{1}{2n^m} + \frac{1+(-1)^p}{2n^q}\zeta(p) - (-1)^p\sum_{k=1}^{\lfloor \frac{p}{2} \rfloor}\binom{m-2k-1}{p-2k}\frac{\zeta(2k)}{n^{m-2k}}.$$

Finally the residue of the pole of order $m+1$ at 0 is found to be

$$\frac{(-1)^p}{2}\binom{m-1}{q}\zeta(m) + (-1)^{p+1}\sum_{k=1}^{\lfloor \frac{q}{2} \rfloor}\binom{m-2k-1}{p-1}\zeta(2k)\zeta(m-2k).$$

Summing these three contributions yields the statement of the theorem. □

For even weights, a modified form of the identity holds, but without any linear Euler sum occurring. This gives back well-known nonlinear relations between zeta values at even arguments. In this case of even weight w, there also exist relations between linear sums. The kernels

$$\xi_j(s) = [\psi^{(j)}(-s)]^2 \tag{20}$$

applied to s^{-q} yield further relations. (For $j = 1, 2$, the general summation formulæ are given in (S_4) and (S_5) of Fig 1.) When specialized to $r(s) = s^{-q}$, the kernel ξ_j yields linear relations between

$$S_{2j+1,q}, S_{2j,q+1}, \ldots, S_{j+1,q+j} \tag{21}$$

and polynomials in zeta values that are of a shape similar to the Euler-Nielsen relations. This gives the reductions

$$S_{3,q} \mapsto S_{2,q+1}, \quad S_{5,q} \mapsto \{S_{2,q+3}, S_{4,q+1}\}, \quad S_{7,q} \mapsto \{S_{2,q+5}, S_{4,q+3}, S_{6,q+1}\},$$

and so on. Such relations are to be complemented by the symmetry relations (5).

Identity (c) of Eq. (3) is an evaluation that is typical of odd weight identities. For the exceptional even weights $\{4,6\}$, the symmetry relations give $S_{2,2}$ and $S_{3,3}$, whence, by $(S_4),(S_5)$ of Fig 1, all linear sums,

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\frac{7}{4}\zeta(4),\quad \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}=\frac{1}{2}\zeta^2(3)+\frac{1}{2}\zeta(6),\quad \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta(3)^2-\frac{1}{3}\zeta(6).$$

For the next even weights, we obtain relations from which it results, again in conjunction with the symmetry relations, that the sets

$$\{S_{2,6}\},\quad \{S_{2,8}\},\quad \{S_{2,10}\},\quad \{S_{2,12},S_{4,10}\}$$

are sufficient to express linearly all linear sums of weights $8,10,12,14$ (modulo zeta values). For instance, we have the relations:

$$\begin{aligned}
5\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^5} &= -\frac{21}{4}\zeta(8)+10\zeta(3)\zeta(5),\\
7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8}+2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^7} &= -\frac{33}{2}\zeta(10)+14\zeta(3)\zeta(7)+8\zeta(5)^2,\\
7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8}-2\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^6} &= -\frac{227}{10}\zeta(10)+14\zeta(3)\zeta(7)+10\zeta(5)^2.
\end{aligned}$$

Zagier [16], by means of an analogy with the theory of modular forms, and Borwein *et al.* [4], by exploiting directly the Euler-Nielsen relations, have shown that the linear relations of even weight determine all but $\lfloor\frac{w-2}{6}\rfloor$ of the linear Euler sums that are thus considered to be "new" constants.

Note on the choice of kernels. The kernels are rather directly related to the quantities subject to summation. As we have seen, the residues of $(\psi(-s)+\gamma)^2$ generate the harmonic numbers, so that sums involving H_n should be represented by integrals involving this kernel, in accordance with (S_3) of Fig. 1. The kernel $\psi'(-s)^2$ similarly introduces $H_n^{(2)}$ and $H_n^{(3)}$ and thus generates relation (S_4) that involves two types of harmonic numbers. Furthermore, by combining formulæ for $r(s)$ and $r(-s)$, the terms involving $H_n^{(3)}$ disappear when $r(s)$ is an odd function; the use of $\pi\cot\pi s$ as replacement for one factor of $\psi'(-s)$ precisely has the effect of achieving such a combination. Thus a sum like $\sum H_n^{(2)}r(n)$ becomes reducible when $r(s)$ is an odd function. Similar observations dictate the choice of kernels throughout this paper as is illustrated by Fig. 1, 2.

4. Quadratic Euler sums

Starting from an observation of Au-Yeung that

$$S_{1^2,2}=\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^2}=\frac{17}{4}\zeta(4),$$

Borwein *et al.* [4] have given a general reduction of the quadratic sums $S_{1^2,q}$ to double sums, which in turn entails a complete evaluation in terms of single zeta values for odd weight. These sums are closely related to derivatives of the Eulerian beta integral. We show here a direct derivation of the reductions by means of ψ kernels that provides in passing general summatory formulæ for sums involving $(H_n)^2$. (See also Fig 1 and Section 8.)

$$
\begin{array}{lll}
(S_1) & \displaystyle\sum_{n=1}^{\infty} r(n) & = \; -\mathcal{R}[r(s)(\psi(-s)+\gamma)] \\
(S_2) & \displaystyle 2\sum_{n=1}^{\infty} r_0(n) & = \; -\mathcal{R}[r_0(s)\pi\cot\pi s] \\
(S_3) & \displaystyle 2\sum_{n=1}^{\infty} r(n)H_n + \sum_{n=1}^{\infty} r'(n) & = \; -\mathcal{R}[r(s)(\psi(-s)+\gamma)^2] \\
(S_4) & \displaystyle -4\sum_{n=1}^{\infty} H_n^{(3)} r(n) + 2\sum_{n=1}^{\infty} H_n^{(2)} r'(n) & \\
& \displaystyle \quad + \sum_{n=1}^{\infty}\left(4\zeta(3)r(n) + 2\zeta(2)r'(n) + \frac{1}{6}r'''(n)\right) & = \; -\mathcal{R}[r(s)(\psi'(-s))^2] \\
(S_5) & \displaystyle 48\sum_{n=1}^{\infty} H_n^{(5)} r(n) - 24\sum_{n=1}^{\infty} H_n^{(4)} r'(n) + 4\sum_{n=1}^{\infty} H_n^{(3)} r''(n) & \\
& \displaystyle \quad + \sum_{n=1}^{\infty}\left(-48\zeta(5)r(n) - 24\zeta(4)r'(n) - 4\zeta(3)r''(n) + \frac{1}{30}r^{(v)}(n)\right) & = \; -\mathcal{R}[r(s)(\psi''(-s))^2] \\
(S_6) & \displaystyle 2\sum_{n=1}^{\infty} H_n^{(2)} r_1(n) + \sum_{n=1}^{\infty}\left(\frac{1}{2}r_1''(n) - 2\zeta(2)r_1(n) - \frac{r_1(n)}{n^2}\right) & = \; -\mathcal{R}[r_1(s)\psi'(-s)\pi\cot(\pi s)] \\
(S_7) & \displaystyle 3\sum_{n=1}^{\infty} r(n)(H_n)^2 - 3\sum_{n=1}^{\infty} r(n)H_n^{(2)} + 3\sum_{n=1}^{\infty} H_n r'(n) & \\
& \displaystyle \quad + \sum_{n=1}^{\infty}\left(\frac{1}{2}r''(n) - 3r(n)\zeta(2)\right) & = \; -\mathcal{R}[r(s)(\psi(-s)+\gamma)^3]
\end{array}
$$

FIGURE 1. General summatory formulæ resulting from kernels (last column) that are polynomial forms in ψ functions. There $r(s), r_0(s), r_1(s)$ denote rational functions that satisfy the conditions (16), with additionally $r_0(s)$ even and $r_1(s)$ odd. Cubic formulæ are given in the proof of Theorem 5.

Theorem 3 (Borwein *et al.*). *For all weights, the quadratic sums $S_{1^2,q}$ reduce to linear sums and polynomials in zeta values:*

$$S_{1^2,q} - S_{2,q} = qS_{1,q+1} - \frac{q(q+1)}{6}\zeta(q+2) + \zeta(2)\zeta(q).$$

Proof. The proof is based on the cubic kernel

$$\xi(s) = (\psi(-s)+\gamma)^3$$

and the usual residue computation. When applied to an arbitrary rational function $r(s)$ satisfying (16), it yields a summatory formula that is tabulated as (S_7) in Fig. 1. The specialization to $r(s) = s^{-q}$ gives the statement. □

In Theorem 3, for even weights ≥ 8, only $S_{1,q+1}$ reduces to zeta values. For odd weights, both $S_{1,q+1}$ and $S_{2,q}$ reduce to zeta values, hence a complete evaluation. We have, for small odd weight,

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3), \qquad \sum_{n=1}^{\infty}\frac{(H_n)^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4),$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^7} = \frac{55}{6}\zeta(9) - \zeta(2)\zeta(7) - \frac{7}{2}\zeta(3)\zeta(6) - \frac{5}{2}\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3,$$

Kernel	*Reduction*	Order	
$(\psi(-s)+\gamma)^2$	$S_{1,r}$	1	Reduction, all r (Thm. 1)
$\psi^{(p-1)}(-s)\pi\cot\pi s$	$S_{p,q}$	1	Reduction, odd weight $p+q$ (Thm. 2)
$(\psi^{(j)}(-s))^2$	$S_{2j+1,q},\ldots,S_{j+1,q+j}$	1	Relations, even weight (Eq. (20,21))
$(\psi(-s)+\gamma)^3$	$S_{1^2,q}-S_{2,q}$	2	Reduction, all weight
$\psi^{(i-1)}(-s)\psi^{(j-1)}(-s)\pi\cot\pi s$	$S_{ij,k}\mapsto\{S_{a,b}\}$	2	Reduction of order, even weight (Thm. 4)
$(\psi(-s)+\gamma)^4$	$S_{1^3,q}-3S_{1\,2,q}$	3	Reduction, all weight (Thm. 5)

FIGURE 2. A summary of kernels and the corresponding reductions.

and for small even weight,

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^6}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}=\frac{91}{12}\zeta(8)-8\zeta(3)\zeta(5)+\zeta(2)\zeta(3)^2,$$
$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^8}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8}=\frac{473}{40}\zeta(10)-10\zeta(3)\zeta(7)-5\zeta(5)^2+\zeta(4)\zeta(3)^2+2\zeta(2)\zeta(3)\zeta(5),$$

with the exceptional evaluations for weights $\{4,6\}$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^2}=\frac{17}{4}\zeta(4),\qquad \sum_{n=1}^{\infty}\frac{(H_n)^2}{n^4}=\frac{97}{24}\zeta(6)-2\zeta(3)^2. \tag{22}$$

The sum $S_{1^2,q}$ is also related to the triple zeta function $\zeta(1,1,q)$ since

$$S_{1^2,q}-S_{q,2}=2\zeta(1,1,q)-\zeta(q+2)+S_{q+1,1},$$

as shown by an elementary computation. Thus, the statement is equivalent to a reduction of $\zeta(1,1,q)$ to double zetas.

General quadratic sums. A more general reduction results from the kernel

$$\frac{\psi^{(p_1-1)}(-s)}{(p_1-1)!}\frac{\psi^{(p_2-1)}(-s)}{(p_2-1)!}\pi\cot\pi s \tag{23}$$

but it involves a parity restriction on the weight because of its trigonometric factor.

Theorem 4. *If p_1+p_2+q is even, and $p_1>1$, $p_2>1$, $q>1$, the quadratic sums*

$$S_{p_1p_2,q}=\sum_{n\ge1}\frac{H_n^{(p_1)}H_n^{(p_2)}}{n^q}$$

are reducible to linear sums. We have

$$[(-1)^{p_1+p_2+q}+1]S_{p_1p_2,q}+A-2(-1)^{p_2}B-2(-1)^{p_1}C-2D+E-2F=0,$$

where

$$
\begin{aligned}
A &= (-1)^{p_1+p_2}\zeta(p_1)\zeta(p_2)\zeta(q)+(-1)^{p_1}\zeta(p_1)S_{p_2,q}+(-1)^{p_2}\zeta(p_2)S_{p_1,q}\\
B &= \sum_{i+j+2k=p_1}(-1)^j\binom{j+q-1}{q-1}\binom{p_2+i-1}{p_2-1}[(-1)^{p_2+i}S_{p_2+i,q+j}+\zeta(p_2+i)\zeta(q+j)]\zeta(2k)\\
C &= \sum_{i+j+2k=p_2}(-1)^j\binom{j+q-1}{q-1}\binom{p_1+i-1}{p_1-1}[(-1)^{p_1+i}S_{p_1+i,q+j}+\zeta(p_1+i)\zeta(q+j)]\zeta(2k)\\
D &= \sum_{j+2k=p_1+p_2}(-1)^j\binom{j+q-1}{q-1}\zeta(2k)\zeta(q+j)\\
E &= (-1)^{p_1+p_2+q}\,[-S_{p_1,p_2+q}-S_{p_2,p_1+q}-\zeta(p_1)S_{p_2,q}-\zeta(p_2)S_{p_1,q}\\
&\qquad +\zeta(p_1+p_2+q)+\zeta(p_1+q)\zeta(p_2)+\zeta(p_2+q)\zeta(p_1)+\zeta(p_1)\zeta(p_2)\zeta(q)]\\
F &= \zeta(p_1+p_2+q)\\
&\quad +(-1)^{p_2}\sum_{i+2k=p_1+q}\binom{p_2+i-1}{p_2-1}\zeta(p_2+i)\zeta(2k)\\
&\quad +(-1)^{p_1}\sum_{i+2k=p_2+q}\binom{p_1+i-1}{p_1-1}\zeta(p_1+i)\zeta(2k)\\
&\quad +(-1)^{p_1+p_2}\sum_{i_1+i_2+2k=q}\binom{p_1+i_1-1}{p_1-1}\binom{p_2+i_2-1}{p_2-1}\zeta(p_1+i_1)\zeta(p_2+i_2)\zeta(2k).
\end{aligned}
$$

The sums are over all indices ≥ 0. *The value* $\zeta(0)=-\frac{1}{2}$ *should be used and* $\zeta(1)$ *should be replaced by* 0 *whenever it occurs.*

Proof. Use the kernel of (23). The quantity F represents

$$-\mathcal{R}\left[s^{-q}\frac{\psi^{(p_1-1)}(-s)}{(p_1-1)!}\,\frac{\psi^{(p_2-1)}(-s)}{(p_2-1)!}\pi\cot\pi s\right],$$

that is estimated as a Taylor coefficient. The other quantities represent combined contributions of the poles at $s=\pm n$. $\square$

A similar (slightly simpler!) expression holds in the case when either $i=1$ or j=1, in which case one should replace $\psi^{(0)}(-s)$ by $\psi(-s)+\gamma$.

As is well known, the multiple zeta functions satisfy *shuffle* relations that generalize the symmetry relation (6). For instance,

$$\zeta(a)\zeta(b,c)=\zeta(a,b,c)+\zeta(a+b,c)+\zeta(b,a,c)+\zeta(b,a+c)+\zeta(b,c,a) \tag{24}$$

is valid for $a>1$ and $c>1$, as seen by considering all ways of interlacing the vector arguments (a) and (b,c). The conjunction of the theorem and shuffle relations, provides a simple proof of "half" of the main result of Borwein and Girgensohn [5] according to which all triple zeta values of even weight are reducible to double zeta values. The reductions obtained are in addition explicit double convolutions of simple and double zeta values.

Corollary 2 (Borwein and Girgensohn). *For* $c>1$, *triple zeta values* $\zeta(a,b,c)$ *whose weight* $a+b+c$ *is even are reducible to double zeta values or equivalently to linear Euler sums.*

Proof. It suffices to consider the trivially modified quadratic sums

$$\begin{aligned} T(i,j,k) &:= \sum_{n=1}^{\infty} H_{n-1}^{(i)} H_{n-1}^{(j)} \frac{1}{n^k} \\ &= S_{ij,k} - S_{j,k+i} - S_{i,k+j} + \zeta(i+j+k) \\ &= \zeta(i,j,k) + S_{i+j,k} + \zeta(j,i,k). \end{aligned}$$

Assume first that $j > 1$; $k > 1$ is granted. Then, from the shuffle relations with $a = k, b = i, c = j$, we find

$$\begin{aligned} \zeta(i,j,k) &= \zeta(k)\zeta(i,j) - \zeta(k+i,j) - \zeta(i,k+j) - [\zeta(k,i,j) + \zeta(i,k,j)] \\ &= \zeta(k)\zeta(i,j) - \zeta(k+i,j) - \zeta(i,k+j) - [T(i,j,k) - \zeta(i+j,k) - \zeta(i+j+k)]. \end{aligned}$$

The dual case when $i > 1$ is treated by the substitution $a = k, b = j, c = i$. If both i and j equal 1, then the reduction is attained by the computation of $S_{1^2,k}$. □

It is believed that no reduction holds in general for triple zetas of odd weights [5]. Actually, starting at (odd) weight 11, it seems that $\zeta(5,3,3)$ is independent of single zeta values. (Such properties can be approached heuristically by means of linear integer dependency algorithms based on lattice reduction or related techniques.) However, for the exceptional odd weights $\{5,7,9\}$, all triple zeta values are now known to be reducible to polynomials in single zetas: this is the other "half" of the main result of Borwein and Girgensohn in [5] already referred to that we extend a little bit further in Section 6. An indirect consequence to be discussed in the next section is the reduction of the cubic sums $S_{1^3,q}$ corresponding to special quadruple zeta values.

5. Cubic and higher order Euler sums

For higher degree sums, like the cubic

$$S_{1^3,q} := \sum_{n=1}^{\infty} \frac{(H_n)^3}{n^q}, \tag{25}$$

it is natural to consider the kernels $(\psi(-s)+\gamma)^4$ and $(\psi(-s)+\gamma)^3 \pi \cot \pi s$. Cross products start to proliferate but the relations obtained at the previous steps help reduce many of the sums.

Theorem 5. *(i) For odd weights, the cubic combination $S_{1^3,q} - 3S_{1\,2,q}$ is expressible in terms of zeta values.*

(ii) For even weights, both $S_{1^3,q}$ and $S_{1\,2,q}$ are reducible to $S_{2,q+1}$ and to polynomials in zeta values.

Proof. Let $r(s), r_1(s)$ satisfy the conditions (16) and $r_1(s)$ be additionally odd. Then a direct residue computation gives

$$4\sum_{n=1}^{\infty} r(n)[(H_n)^3 - 3H_nH_n^{(2)}] + 6\sum_{n=1}^{\infty} r'(n)(H_n)^2 + \sum_{n=1}^{\infty} 4[H_n^{(3)} - 3\zeta(2)H_n - \zeta(3)]r(n)$$
$$-4\sum_{n=1}^{\infty}[H_n^{(2)} + \zeta(2)]r'(n) + 2H_n r''(n) + \frac{r'''(n)}{6} = -\mathcal{R}\left[(\psi(-s)+\gamma)^4 r(s)\right],$$

$$-6\sum_{n=1}^{\infty} r_1(n)H_nH_n^{(2)} + 3\sum_{n=1}^{\infty}(H_n)^2[\frac{r_1(n)}{n} + r_1'(n)]$$
$$+3\sum_{n=1}^{\infty}[H_n^{(3)} - (4\zeta(2) + \frac{1}{n^2})H_n - \zeta(3) + \frac{1}{3n^3}]r_1(n)$$
$$-\sum_{n=1}^{\infty}[3H_n^{(2)} + 5\zeta(2)]r_1'(n) + \frac{3}{2}H_n r_1''(n) + \frac{r_1'''(n)}{6} = -\mathcal{R}\left[(\psi(-s)+\gamma)^3\pi\cot(\pi s)r_1(s)\right].$$

These formulæ complement those of Fig. 1.

Instantiating the first identity to $r(s) = s^{-q}$ with even q and appealing to relations (S_4), (S_6) and (S_7) yields the first part of the theorem. The second identity is an explicit version of the quadratic reductions discussed in the previous section; it permits to dispose of the sum $S_{1\,2,q}$ that reduces to the linear sums $S_{2,q+1}$ for even weight. Instantiating it to $r(s) = s^{-q}$ with odd q yields the second part of the theorem. $\square$

For even weight, we thus have an infinite collection of explicit reductions amongst which

$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{n^5} - \frac{11}{4}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6} = \frac{469}{32}\zeta(8) - 16\zeta(3)\zeta(5) + \frac{3}{2}\zeta(2)\zeta(3)^2$$
$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{n^7} - \frac{13}{4}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8} = \frac{561}{20}\zeta(10) - \frac{47}{4}\zeta(5)^2 - \frac{49}{2}\zeta(7)\zeta(3) + 3\zeta(2)\zeta(3)\zeta(5) + \frac{15}{4}\zeta(3)^2\zeta(4)$$
$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{n^9} - \frac{15}{4}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{10}} = \frac{1060345}{22112}\zeta(12) - 33\zeta(5)\zeta(7) - 35\zeta(3)\zeta(9) - \frac{1}{4}\zeta(3)^4 + \frac{3}{2}\zeta(2)\zeta(5)^2$$
$$+\frac{21}{4}\zeta(3)^2\zeta(6) + \frac{15}{2}\zeta(3)\zeta(4)\zeta(5) + 3\zeta(2)\zeta(3)\zeta(7),$$

were given as conjectures in [1, Table 4].

Corollary 3. *The cubic sums $S_{1^3,q}$ of weights $\{5,6,7,9\}$ are reducible to zeta values,*

$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{(n+1)^2} = \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3)$$
$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{(n+1)^3} = -\frac{33}{16}\zeta(6) + 2\zeta(3)^2$$
$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{(n+1)^4} = \frac{119}{16}\zeta(7) - \frac{33}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5)$$
$$\sum_{n=1}^{\infty}\frac{(H_n)^3}{(n+1)^6} = \frac{197}{24}\zeta(9) - \frac{33}{4}\zeta(4)\zeta(5) - \frac{37}{8}\zeta(3)\zeta(6) + \zeta(3)^3 + 3\zeta(2)\zeta(7).$$

(The forms given are those of [1].)

Proof. We only indicate briefly the chain of reductions. For weight 6, this results from the evaluation of $S_{2,4}$ in (22). For weight 5, the evaluation follows from Hoffman's [9] complete reduction of multiple zetas in the case of all weights ≤ 6. For the odd weights $\{7,9\}$, the reduction follows from the Borwein-Girgensohn result after which triple zetas are reducible to double and single zetas for all weights ≤ 10. Alternatively, one may use reduction by any maximal system of relations presented in Section 6. □

Higher degree Euler sums. Linear Euler sums reduce to zeta values in the case of an odd weight, while quadratic Euler sums reduce to linear sums (double zeta values) in the case of an even weight. We prove here a result to the effect that such reductions of order are general.

Theorem 6. *(i) For odd weight $w = i + j + k + \ell$, all cubic sums $S_{ijk,\ell}$ reduce to combinations of Euler sums of order at most 2.*

(ii) More generally, a nonlinear Euler sum

$$S_{i_1 i_2 \cdots i_r, q}$$

reduces to a combination of sums of lower orders whenever the weight $i_1 + i_2 + \cdots + i_r + q$ and the order r are of the same parity.

Proof. We start with the case of cubic sums and adopt the kernel

$$\xi_{i,j,k} = \frac{1}{(i-1)!(j-1)!(k-1)!}\psi^{(i-1)}(-s)\psi^{(j-1)}(-s)\psi^{(k-1)}(-s)\pi\cot\pi s$$

that is applied to $r(s) = s^{-\ell}$. The expansion at $s = m$,

$$\frac{1}{(i-1)!}\psi^{(i-1)}(-s) = \frac{1}{(s-m)^i} + H_m^{(i)} + (-1)^i\zeta(i) + \cdots,$$

implies that the sum of residues at positive integers is of the form $S_{ijk,\ell}+T$, where T is a combination of quadratic sums. The expansion at $s = -m$,

$$\frac{1}{(i-1)!}\psi^{(i-1)}(-s) = (-1)^{i-1}[H_{m-1}^{(i)} - \zeta(i)] + \cdots,$$

implies that the sum of residues at negative integers is of the form $(-1)^{i+j+k+\ell-3}S_{ijk,\ell} + U$, where U is a combination of quadratic sums. We thus have a reduction of order whenever the weight is odd.

The general case follows along the very same lines. □

Broadhurst has made a conjecture (presented in [5]) of a shape similar to our statement but concerning multiple zeta values instead. In the case of quadratic sums, we have at least seen that the shuffle relations entail a corresponding reduction for all triple zeta values. It does not seem that Broadhurst's conjecture can be deduced, even partially, from our theorem.

6. Models of Euler sum identities

Various approaches have been developed for Euler sums evaluations. We discuss here general methods and leave aside methods based on definite integrals and polylogarithms of which De Doelder's paper [7] is typical. Our purpose here is to obtain complete models for low weights and at the same time examine the power of various frameworks proposed, including the residue method.

Shuffle relations (Σ)**.** These are relations that generalize the symmetry relation (shuffle of order 2) of Eq. (6) and the particular shuffle of order 3 of (24). Consideration of the product of two multiple zeta functions $\zeta(\mathbf{u}), \zeta(\mathbf{v})$, with $\mathbf{u}, \mathbf{v}$ denoting arbitrary vectors of integers, gives the relation,

$$\zeta(\mathbf{u}) \cdot \zeta(\mathbf{v}) = \sum_{\mathbf{w} \in (\mathbf{u} \,ш\, \mathbf{v})} \zeta(\mathbf{w}). \tag{26}$$

There $(\mathbf{u} \,ш\, \mathbf{v})$ is the *shuffle* of vectors $\mathbf{u}, \mathbf{v}$ that is a set of vectors defined recursively by

$$(a \cdot \mathbf{u}) \,ш\, (b \cdot \mathbf{v}) = a \cdot [\mathbf{u} \,ш\, (b \cdot \mathbf{v})] \;\cup\; b \cdot [(a \cdot \mathbf{u}) \,ш\, \mathbf{v}] \;\cup\; (a + b) \cdot (\mathbf{u} \,ш\, \mathbf{v}),$$

where the dot operation is the concatenation of vectors (extended to sets in the usual way) and all operations are taken in the sense of multisets so as to preserve multiplicities.

Equation (26) simply expresses all possible interlacings of indices when a product is expanded by distributivity. The shuffle relations are similar to symmetric function identities studied by Hoffman [9] and, as noted by Zagier [16], they imply that the linear space spanned by the multiple zeta values forms a ring.

Duality (Δ)**.** Duality is a surprising property first conjectured by Hoffman [9] and proved by Zagier in [16] upon a suggestion of Kontsevich. It is expressed by means of an encoding by binary vectors of multiple zeta values: given a vector $\mathbf{u} = (u_1, \ldots, u_k)$, its encoding is

$$\beta(u_1, u_2, \ldots, u_k) := 10^{u_1 - 1} 10^{u_2 - 1} \cdots 10^{u_k - 1},$$

where 0^k means 0 repeated k times. We then introduce the quantities

$$H(U) := \zeta(\beta^{(-1)} U),$$

that are defined for all binary vectors starting with a 1 and ending with a 0. Define the reverse-complement of a binary vector $U = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_\ell$ as $U^\star = \overline{\varepsilon}_\ell \overline{\varepsilon}_{\ell-1} \cdots \overline{\varepsilon}_1$, where $\overline{\varepsilon} = 1 - \varepsilon$. Then Hoffman's duality principle states that

$$H(U) = H(U^\star). \tag{27}$$

This relation groups the multiple zetas into equal pairs and, for instance, implies that

$$\zeta(2, 3, 4) = H(101001000) = H(111011010) = \zeta(1, 1, 2, 1, 2, 2).$$

The proof sketched in [16] is based on the multiple integral representation

$$H(\varepsilon_1, \ldots, \varepsilon_k) = \int \cdots \int_{0 < t_1 < \cdots < t_k < 1} d_{\varepsilon_1} t_1 \cdots d_{\varepsilon_k} t_k, \qquad d_0 t = \frac{dt}{t}, \quad d_1 t = \frac{dt}{1 - t},$$

and on the change of variables $u_j = 1 - t_j$.

Partial fraction expansions (Π)**.** The Euler-Nielsen method, of which an idea was given at the beginning of Section 3 applies to double zetas [13], and, as established by Markett [12] and Borwein-Girgensohn [5], it can be extended to triple zetas. We let Π_2 and Π_3 denote the linear relations that derive from this mechanism in the case of zetas of multiplicities 2 and 3.

Residue relations (R). We have designed a program in the computer algebra system MAPLE that computes relations on Euler sums that result from any kernel that is a polynomial form in ψ functions and their derivatives. This permits to investigate exhaustively the relations deriving from the residue method applied to Euler sums of a fixed given weight.

With the help of the MAPLE system, we have investigated the dimension of the spaces of linear relations that result from any combination of the rules $\Sigma, \Delta, \Pi_2, \Pi_3, R$ for all weights till 10. This can be viewed as a supplement to Hoffman's investigations who obtained a complete basis of relations between multiple zetas for weights ≤ 6.

First, the linear relations implied by the rules $\Sigma, \Delta, \Pi_2, \Pi_3, R$ take place in the space of products of multiple zetas with total weight w, of which there are p_w, given by

$$\begin{aligned} \sum_{w=2}^{\infty} p_w z^w &= \prod_{j=2}^{\infty} \frac{1}{(1-z^j)^{2^{j-2}}} \\ &= 1 + z^2 + 2\,z^3 + 5\,z^4 + 10\,z^5 + 24\,z^6 + 50\,z^7 + 114\,z^8 + 246\,z^9 + 546\,z^{10} + \cdots . \end{aligned}$$

The shuffle relations reduce these products into linear combinations of multiple zetas of weight w, forming a space whose dimension is 2^{w-2}. There are E_w distinct Euler sums, where

$$\begin{aligned} \sum_{w=2}^{\infty} E_w z^w &= \frac{z^2}{1-z} \prod_{j=1}^{\infty} \frac{1}{1-z^j} \\ &= z^2 + 2\,z^3 + 4\,z^4 + 7\,z^5 + 12\,z^6 + 19\,z^7 + 30\,z^8 + 45\,z^9 + 67\,z^{10} + 97\,z^{11} + \cdots , \end{aligned}$$

and standard estimates on the number of partitions imply that $E_w = e^{O(\sqrt{w})}$.

We seek reductions of Euler sums into linear combinations of monomials in single zeta values whose number μ_w satisfies

$$\begin{aligned} \sum_{w=0}^{\infty} \mu_w z^w &= \frac{1}{1-z^2} \prod_{j=1}^{\infty} \frac{1}{1-z^{2j+1}} \\ &= 1 + z^2 + z^3 + z^4 + 2\,z^5 + 2\,z^6 + 3\,z^7 + 3\,z^8 + 5\,z^9 + 5\,z^{10} + 7\,z^{11} + 8\,z^{12} + \cdots . \end{aligned}$$

The growth order of μ_w is again $e^{O(\sqrt{w})}$, though with a smaller exponential rate than E_w. These μ_w (presumably $\mathbb{Q}$-linearly independent) monomials span the space of "closed-form" expressions.

Thus, the numbers of multiple zeta forms, Euler sums, and polyzeta forms satisfy

$$2^{w-2} \gg E_w \gg \mu_w .$$

Therefore, one should not expect on these grounds all multiple zetas nor even all Euler sums to reduce to combinations of zeta monomials. In other words, closed form is exceptional for an Euler sum.

Zagier has conducted extensive numerical computations of multiple zeta values of all weights till 12 inclusive and has examined the apparent $\mathbb{Q}$-linear dependencies that result. Based on these computations and other algebraic arguments, he conjectures that the dimension d_w is given by the recurrence

$$d_2 = d_3 = d_4 = 1, \quad d_w = d_{w-2} + d_{w-3},$$

so that

$$\begin{aligned}\sum_{w=2}^{\infty} d_w z^w &= \frac{1}{1-z^2-z^3} \\ &= 1+z^2+z^3+z^4+2\,z^5+2\,z^6+3\,z^7+4\,z^8+5\,z^9+7\,z^{10}+9\,z^{11}+12\,z^{12}+\cdots.\end{aligned}$$

The growth of d_w is of the approximate form $d_w \approx 1.32471^w$.

Thus, modulo Zagier's conjecture, the actual dimension of the $\mathbb{Q}$-linear space of multiple zeta values lies somewhere in between the (large) number 2^{w-2} of multiple zetas and the (small) number μ_w of closed-form monomials. What is remarkable, however, is that there is almost coincidence of d_w and μ_w for weights < 10, the difference $d_8 - \mu_8 = 1$ being simply accounted by the occurrence of the (probably) irreducible $S_{2,6}$. Based on our program, we have verified the reductions implied by Zagier's conjecture for all weights till 9 inclusive. (We do not claim much originality for the result that follows as it is largely a verification based on technologies due to Hoffman, Zagier, Markett, and Borwein-Girgensohn.)

Theorem 7 (Folklore). *All multiple zetas of weight ≤ 9 are reducible to $\mathbb{Q}$-linear combinations of single zeta monomials with the addition of $\{S_{2,6}\}$ for weight 8.*

Proof. Solve the linear systems deriving from the shuffle relations Σ, duality Δ, partial fractions Π_2 and Π_3, and residues R. □

Corollary 4. *All Euler sums of the form $S_{1^p,q}$ for weights $p+q \in \{3,4,5,6,7,9\}$ are expressible polynomially in terms of zeta values. For weight 8, all such sums are the sum of a polynomial in zeta values and a rational multiple of $S_{2,6}$.*

This corollary provides a justification of identities discovered experimentally by Bailey *et al.* [1].

In passing, the computations underlying Theorem 7 permit to delineate the power of various reduction principles. First, duality reduces by about a half the number of independent multiple zetas to be considered since it provides a number δ_w of nontrivial linear equalities that satisfies $\delta_w = 2^{w-3}$ when w is odd and $\delta_w = 2^{w-3} - 2^{w/2-2}$ when w is even. Next, the shuffle relations reduce all the products of multiple zetas to linear combinations of multiple zetas. Besides, the shuffle relations induce linear relations on multiple zetas. For instance, since $\zeta(1,2) = \zeta(3)$, the products of these by $\zeta(2)$ once expanded by the shuffle relations yield:

$$\zeta(2,1,2) + 2\zeta(1,2,2) + \zeta(1,4) - \zeta(2,3) - \zeta(5) = 0.$$

The Nielsen relations Π_2 appear to provide $\lfloor w/2 \rfloor$ independent linear relations of weight w, which is not much. Also, for odd weight, these relations are implied by the residue relations R as expressed by Theorem 2. The Markett relations Π_3 appear to induce $O(w^2)$ independent linear relations of weight w. In Table 1, we give the dimension of the vector space of linear relations induced by the rule Σ; we also give the dimension of the linear relations induced by Π_2, Π_3, Δ, and R once linearized by the shuffle relations. The total dimension of the space of relations we get is indicated in the next column. It is to be compared with the value $2^{w-2} - d_w$ implied by Zagier's conjecture. In the last column we indicate which minimal combinations of relations make it possible to generate all the known relations (in conjunction with Σ).

An interesting aspect of the proof of Theorem 7 is that residue relations contribute new relations to the arsenal of currently known methods and permit to attain the limit described by Zagier's conjecture for weights till 9 inclusive. This is demonstrated in the last column of Table 1, where it appears that all 4 relations are necessary to get 123 independent linear relations of weight 9 (since

weight w	Σ	Δ	Π_2	Π_3	R	total	$2^{w-2}-d_w$	
3	0	1	1	0	1	1	1	$(\Pi_2),(\Delta),(R)$
4	0	1	2	1	2	3	3	$(\Pi_2,\Delta),(\Pi_2,\Pi_3),(\Pi_2,R),(\Pi_3,R)$
5	1	4	2	3	5	6	6	$(\Pi_2,\Delta),(\Pi_3,\Delta),(R,\Delta),(\Pi_3,R)$
6	5	6	3	6	10	14	14	$(\Pi_2,\Delta),(\Pi_3,\Delta),(R,\Delta)$
7	12	16	3	10	17	29	29	(Π_3,Δ)
8	31	28	4	15	31	60	60	$(\Pi_3,\Delta),(R,\Delta)$
9	68	64	4	21	45	123	123	(Π_3,Δ,R)
10	151	120	5	27	75	248	249	$(\Pi_3,\Delta),(R,\Delta)$

TABLE 1. Rank of relations vs. weight. Each set of relations generates a vector space of linear relations on the multiple zetas. For each weight, we indicate the dimension of this space, which gives a measure of the power of the relations.

the weight is odd Π_2 is implied by R). For instance, the kernel $[\psi(s)+\gamma]^3\psi'(-s)$ applied to the base function $1/s^5$ induces (after linearizing by the shuffle relations):

$$\begin{aligned}&21\zeta(9)+40\zeta(1,2,6)+40\zeta(1,8)-32\zeta(1,1,7)-72\zeta(2,7)+61\zeta(3,6)+40\zeta(2,1,6)\\&\quad+48\zeta(4,5)-16\zeta(2,2,5)+48\zeta(5,4)+112\zeta(3,3,3)+37\zeta(6,3)-32\zeta(1,1,5,2)+24\zeta(1,6,2)\\&\qquad+16\zeta(1,5,3)+80\zeta(1,1,1,6)-16\zeta(2,5,2)-56\zeta(7,2)=0,\end{aligned}$$

which is not a consequence of the linear relations induced by the partial fraction relations together with duality and the shuffle relations. For weight 10, the last line of Table 1 indicates that the relations $(\Sigma,\Delta,\Pi_2,\Pi_3,R)$ are no longer sufficient to generate all the linear relations implied by Zagier's conjecture.

7. Alternating Euler sums

We now turn to the evaluation of alternating Euler sums by means of contour integrals. There are altogether 4 types of linear sums,

$$(28)\qquad \begin{aligned} S^{++}_{p,q} &= \sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q} & \qquad S^{-+}_{p,q} &= \sum_{n=1}^{\infty}\frac{\overline{H}_n^{(p)}}{n^q}\\ S^{+-}_{p,q} &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(p)}}{n^q} & \qquad S^{--}_{p,q} &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\overline{H}_n^{(p)}}{n^q},\end{aligned}$$

where the $\overline{H}_n^{(p)}$ denote the alternating harmonic numbers,

$$(29)\qquad \overline{H}_n^{(r)}=\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j^r},\qquad \overline{H}_n := H_n^{(1)}=\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}.$$

Clearly $S^{++}_{p,q}$ corresponds to the standard Euler sums defined earlier. Such numbers have been considered by Euler and, after him, by Nielsen and many others.

A natural kernel for the sums of type S^{+-} is a combination of ψ functions and $\pi/\sin\pi s$, as the latter introduces sign alternation. In that case, some parity constraints must however intervene since poles occur at positive *and* negative integers. The other results are best stated in terms of the alternating zeta function,

$$\overline{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s),$$

with $\overline{\zeta}(1) = \log 2$. Alternating harmonic numbers are introduced by the modified ψ function,

$$\begin{aligned} \overline{\psi}(s) &:= \sum_{k=0}^{\infty} \frac{(-1)^k}{s+k} \\ &= \frac{1}{2}\psi\left(\frac{s+1}{2}\right) - \frac{1}{2}\psi\left(\frac{s}{2}\right) \\ &= \frac{1}{s} - \log 2 + \overline{\zeta}(2)s - \overline{\zeta}(3)s^2 + \overline{\zeta}(4)s^3 - \cdots \end{aligned}$$

(this is Nielsen's β function) that satisfies, with n a positive integer,

$$\overline{\psi}(n) = (-1)^n(\overline{H}_{n-1} - \log 2), \quad \overline{\psi}(s) \underset{s\to -n}{=} (-1)^n \left[\frac{1}{(s+n)} + (\overline{H}_n - \log 2) + \cdots\right].$$

The following evaluations are all found in Sitaramachandrarao's paper [14] that contains an exhaustive discussion of sums $S^{\pm\pm}_{1,r}$ together with a thorough bibliography. Here, the identities come out as simple consequences of the process employed earlier for standard Euler sums.

Theorem 8 (Euler, Sitaramachandrarao). *(i) For any weight* $1+q$,

$$2S^{-+}_{1,q} = 2\zeta(q)\log 2 - q\zeta(q+1) + 2\overline{\zeta}(q+1) + \sum_{k=1}^{q} \overline{\zeta}(k)\overline{\zeta}(q-k+1).$$

(ii) In the case of a weight $1+q$ *that is odd,*

$$2S^{+-}_{1,q} = (q+1)\overline{\zeta}(q+1) - \zeta(q+1) - 2\sum_{k=1}^{q/2-1} \overline{\zeta}(2k)\zeta(q+1-2k),$$

$$2S^{--}_{1,q} = 2(\zeta(q)+\overline{\zeta}(q))\log 2 - (q+1)\overline{\zeta}(q+1) + \zeta(q+1) + 2\sum_{k=1}^{q/2-1} \zeta(2k)\overline{\zeta}(q+1-2k).$$

Proof. The result falls as a ripe fruit when using respectively the kernels

$$\overline{\psi}(s)^2, \quad \frac{\pi}{\sin\pi s}(\psi(-s)+\gamma), \quad \overline{\psi}(s)\pi\cot\pi s.$$

In the first case, the sign alternation of the general term disappears because of the squaring of $\overline{\psi}(s)$, so that we get directly $S^{-+}_{1,q}$. In the other cases, two almost identical sums result from the residues at the positive and negative integers, and the combination involves a coefficient of $(1+(-1)^q)$ so that estimates are restricted to the case of an odd q. $\square$

(T_1)	$2\sum_{n=1}^{\infty}(-1)^n r_0(n)$	$=$	$-\mathcal{R}[r_0(s)\frac{\pi}{\sin\pi s}]$
(T_2)	$2\sum_{n=1}^{\infty}\overline{H}_n r(n) - \sum_{n=1}^{\infty}[2\log 2 r(n) + r'(n)]$	$=$	$\mathcal{R}[\overline{\psi}^2(-s)r(s)]$
(T_3)	$2\sum_{n=1}^{\infty}(-1)^n H_n r_0(n) + \sum_{n=1}^{\infty}(-1)^n[r_0'(n) - \frac{r_0(n)}{n}]$	$=$	$-\mathcal{R}[(\psi(-s)+\gamma)\frac{\pi}{\sin\pi s}r_0(s)]$
(T_4)	$2\sum_{n=1}^{\infty}(-1)^n\overline{H}_n r_0(n) - \sum_{n=1}^{\infty}(-1)^n[r_0'(n) + 2\log 2 r_0(n)] - \frac{r_0(n)}{n}$	$=$	$-\mathcal{R}[\overline{\psi}(s)\pi\cot\pi s r_0(s)]$

FIGURE 3. General summatory formulæ for alternating sums. There $r(s), r_0(s)$ denote rational functions that satisfy the conditions (16), with additionally $r_0(s)$ even.

Notice finally that the use of the kernel

$$\overline{\psi}(s)(\psi(-s)+\gamma)$$

permits to relate $S^{+-}_{1,q}$ and $S^{--}_{1,q}$ irrespective of the parity of the weights:

$$S^{--}_{1,q} + (-1)^q S^{+-}_{1,q} = \overline{\zeta}(q)\log 2 - \sum_{i=1}^{q-1}(-1)^i\overline{\zeta}(i)\zeta(q+1-i).$$

In other words, there is a new variety of constants defined by

$$\mu_q = S^{+-}_{1,2q+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^{2q+1}},$$

where[1],

$$\mu_q = \frac{1}{(2q)!}\int_0^1 \frac{\log^{2q}(z)\log(1+z)}{z(1+z)}\,dz.$$

We have from [7, 14]

$$\mu_0 = \frac{1}{2}\zeta(2) - \frac{1}{2}\log^2 2, \quad \mu_1 = -2\,\mathrm{Li}_4(\frac{1}{2}) + \frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\log^2 2 - \frac{1}{12}\log^4 2 - \frac{7}{4}\zeta(3)\log 2,$$

where $\mathrm{Li}_q(z) = \sum_{n=1}^{\infty} z^n n^{-q}$ is the polylogarithm. The constant μ_1 is related to several of Ramanujan's evaluations as well as to the analysis of lattice reduction [6] mentioned in the introduction. Higher order μ's are not known to be related to classical constants.

Following Euler, Nielsen established relations suggesting that alternating sums of odd weight should reduce to polynomials in zeta values augmented with $L = \overline{\zeta}(1) = \log 2$. This approach is developed by Borwein *et al.* [4] who show that the Euler-Nielsen relations can be inverted but stop short of giving explicit formulæ.

Shuffle relations that are analogous to (5),

$$\overline{\zeta}(p)\zeta(q) + \overline{\zeta}(p+q) = S^{-+}_{p,q} + S^{+-}_{q,p}, \quad \overline{\zeta}(p)\overline{\zeta}(q) + \zeta(p+q) = S^{--}_{p,q} + S^{--}_{q,p} \tag{30}$$

reduce the number of quantities to be investigated. Since our interest is in general summatory formulæ, we prefer however to develop an approach from scratch.

[1]In this paper, we have otherwise chosen not to develop connections between Euler sums and definite integrals involving logarithms and polylogarithms.

Theorem 9. *For a weight $w = p + q$ that is odd,*

$$[(-1)^q - (-1)^p]S^{-+}_{p,q} = (-1)^p\overline{\zeta}(p+q) + ((-1)^p - 1)\overline{\zeta}(p)\zeta(q)$$
$$+ 2\sum_{j+2k=p}\binom{q+j-1}{q-1}\zeta(q+j)\overline{\zeta}(2k) + 2(-1)^p\sum_{i+2k=q}\binom{p+i-1}{p-1}(-1)^i\overline{\zeta}(p+i)\overline{\zeta}(2k),$$

$$2S^{+-}_{p,q} = [1-(-1)^p]\zeta(p)\overline{\zeta}(q) + \overline{\zeta}(p+q)$$
$$+ 2\sum_{j+2k=p}\binom{q+j-1}{q-1}(-1)^{j+1}\overline{\zeta}(q+j)\overline{\zeta}(2k) + 2(-1)^p\sum_{i+2k=q}\binom{p+i-1}{p-1}\zeta(p+i)\overline{\zeta}(2k),$$

$$[(-1)^p - (-1)^q]S^{--}_{p,q} = (-1)^{p+1}\zeta(p+q) + (1-(-1)^p)\overline{\zeta}(p)\overline{\zeta}(q)$$
$$+ 2\sum_{j+2k=p}\binom{q+j-1}{q-1}\overline{\zeta}(q+j)\zeta(2k) - 2(-1)^p\sum_{i+2k=q}\binom{p+i-1}{p-1}(-1)^i\overline{\zeta}(p+i)\zeta(2k).$$

Proof. Just use the kernels

$$\frac{1}{(p-1)!}\overline{\psi}^{(p-1)}(s)\frac{\pi}{\sin\pi s}, \qquad \frac{1}{(p-1)!}\psi^{(p-1)}(s)\frac{\pi}{\sin\pi s}, \qquad \frac{1}{(p-1)!}\overline{\psi}^{(p-1)}(s)\pi\cot\pi s.$$

□

8. Exotic sums

The use of kernels involving ψ and its relatives is not just restricted to Euler sums. We have chosen here a random sample of 4 types of "exotic" summatory formulæ pointing the way to extensions of the method.

§1. Consider the family of sums,

$$A_q := \sum_{n=1}^{\infty}\frac{(H_n)^2}{((2n-1)(2n)(2n+1))^q}.$$

We claim that A_q reduces to a polynomial in zeta values and $\log 2$ whenever q is odd.

Let $r(n) = [(2n-1)(2n)(2n+1)]^{-q}$ and take q odd. By Entry (S_7) of Fig. 1, we have a first reduction (modulo values of ψ functions at $\pm\frac{1}{2}$) to $\sum H_n r'(n)$ and $\sum H_n^{(2)} r(n)$. The first sum reduces in all cases; the second sum reduces again since $r(s)$ is assumed to be odd. An instance is then:

$$\begin{aligned} A_3 &\equiv [1]^2\frac{1}{(1\,2\,3)^3} + [1+\frac{1}{2}]^2\frac{1}{(3\,4\,5)^3} + [1+\frac{1}{2}+\frac{1}{3}]^2\frac{1}{(5\,6\,7)^3} + \cdots \\ &= 4\ln^3 2 + \big(\tfrac{7}{8}\zeta(3) - \tfrac{35}{4}\big)\ln^2 2 - \big(\tfrac{45}{32}\zeta(4) + \tfrac{7}{8}\zeta(3) - \tfrac{9}{8}\zeta(2) - 12\big)\ln 2 \\ &\quad + \tfrac{45}{64}\zeta(4) - \tfrac{1}{4}\zeta(2) - \tfrac{3}{32}\zeta(2)\zeta(3) - \tfrac{41}{8}\zeta(3) + \tfrac{17}{32}\zeta(5). \end{aligned}$$

Several related identities but of a simpler form appear in Chap. 9 of Ramanujan's notebooks [2].

This example is typical. A great many evaluations now become of a purely mechanical character. It is then easy to develop general formulæ that systematically turn out to be homogeneous convolution forms in zeta values.

§2. Sums related to Catalan's constant have been discovered by Ramanujan [2] and further explored by Sitaramachandrarao [14]. We offer here the evaluations,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{2n+1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{1}{2}\pi\log 2\\ \sum_{n=0}^{\infty}(-1)^n\frac{H_n}{(2n+1)^3} &= 3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}-\frac{7}{16}\pi\zeta(3)-\frac{1}{16}\pi^3\log 2.\end{aligned} \tag{31}$$

These are obtained as part of an infinite set obtained by applying the kernel

$$(\psi(-s)+\gamma)\frac{\pi}{\sin\pi s}$$

to $(1+2s)^{-q}$, a choice in agreement with Fig. 3. The residues at the negative integers compose the sums to be evaluated, while the residues at the positive integers involve Catalan's constant and its close relatives; the special residue sum only involves the values of ψ functions at 0 and $\pm\frac{1}{2}$. The general summation formula for odd q (omitted) would have a shape closely resembling Thms. 1,8.

Taking even q, we recover well-known identities like

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32},\qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}=\frac{5}{1536}\pi^5.$$

An equivalent approach consists in separating odd and even parts in the sums of (31). This creates integrands involving $(4s+1)^{-1}$ and $(4s+3)^{-1}$. The evaluations must then involve ψ functions taken at $-\frac{1}{4}$ and $-\frac{3}{4}$, and the corresponding values are precisely related to constants of the Catalan class.

In general alternating sums made of harmonic numbers and odd integers must lead to evaluations in the Catalan world, as the discussion above suggests. This observation is also fully consistent with the results of [14].

§3. The use of kernels involving $i=\sqrt{-1}$ in arguments of ψ functions leads to yet another class of summation formulæ. For instance, one has the highly symmetrical,

$$\begin{aligned}\sum_{m,n\geq 1}\frac{1}{m^2(m^2+n^2)} &= \frac{1}{2}\zeta(2)^2 & \sum_{m,n\geq 1}\frac{1}{m^6(m^2+n^2)} &= \zeta(2)\zeta(6)-\frac{1}{2}\zeta(4)^2\\ \sum_{m,n\geq 1}\frac{1}{mn^3(m^2+n^2)} &= \frac{1}{2}\zeta(3)^2 & \sum_{m,n\geq 1}\frac{1}{mn^7(m^2+n^2)} &= \zeta(3)\zeta(7)-\frac{1}{2}\zeta(5)^2,\end{aligned}$$

with a periodicity of exponents modulo 4. These formulæ arise when the kernel

$$(\psi(1+is)+\gamma)(\psi(-s)+\gamma)$$

is applied to $r(s)=s^{-q}$. Zagier discusses a related but "harder" class of sums in [16].

§4. Last, the summation process exemplified by Fig. 1, 3 extends to irrational meromorphic functions provided they remain small on circles (or other large contours) on which the kernel is itself small. In that case, the special residue sum itself becomes an infinite sum, and one has a relation between two types of infinite sums. We just mention as an example the kernel $(\pi\cot\pi s)$ which, when applied to the functions $(\pi\coth\pi s)/s^q$ yields identities like

$$\sum_{n=1}^{\infty}\frac{\coth\pi k}{k^3}=\frac{7}{180}\pi^3,\quad \sum_{n=1}^{\infty}\frac{\coth\pi k}{k^7}=\frac{19}{56700}\pi^7$$

that were discovered by Ramanujan [3].

Acknowledgements. Early discussions with Brigitte Vallée have helped clarify the residue approach to Euler sums computations. We are grateful to Don Zagier for detailed explanations offered to one of us at the "Combinatorics and Physics" meeting, Luminy March 1995 and our current presentation owes much to Zagier's insights. Jon Borwein and Roland Girgensohn shared ideas, preprints, and references throughout the course of this work. Thanks to all.

This work was supported in part by the Long Term Research Project *Alcom-IT* (# 20244) of the European Union.

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